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Dilaton shift under duality and torsion of elliptic complex

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We observe that the ratio of determinants of $2d$ Laplacians which appear in the duality transformation relating two sigma models with abelian isometries can be represented as a torsion of an elliptic (DeRham) complex. As a result, this ratio can be computed exactly and is given by the exponential of a local functional of $2d$ metric and target space metric. In this way the well known dilaton shift under duality is reproduced. We also present the exact computation of the determinant which appears in the duality transformation in the path integral.

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1. Introduction

Duality transformations in two dimensional sigma models with abelian isometries have recently attracted much attention in connection with string theory (see e.g. [1-4] and refs. there).[1] As is well known [1,6,3], the duality transformation which inverts the metric in the direction of an isometry should also include a shift of the dilaton field (coupled to the curvature of the $2d$ metric). Let us consider a sigma model

$$I = \frac{1}{4\pi} \int d^2z\sqrt{g}\ [(G_{\mu\nu} + B_{\mu\nu})(g^{ab} + i\epsilon^{ab})\partial_a x^\mu \partial_b x^\nu \ + \ R\phi\ ] \ , \tag{1}$$

which is invariant under a global isometry $\delta x^\mu = \epsilon K^\mu$. Then K^μ must be a Killing vector of $G_{\mu\nu}$ and the Lie derivatives of $B_{\mu\nu}$ (and ϕ) must vanish up to a gauge transformation term. In general, one can choose local coordinates $\{x^\mu\} = \{x^1 \equiv y \ , \ x^i\}$ such that $G_{\mu\nu}$, $B_{\mu\nu}$ and ϕ are independent of the coordinate y which is shifted by the isometry. Then the action (1) is at most quadratic in $y(z)$. To obtain the dual model one replaces $\partial_a y$ by a "momentum" field p_a adding the constraint term $\sim \tilde{y}\epsilon^{ab}(\partial_a p_b - \partial_a p_b)$, where $\tilde{y}$ is a Lagrange multiplier. By formally doing the gaussian integral over p_a one finds an action (dual action $\tilde{I}$) for $\{\tilde{x}^\mu\} = \{\tilde{x}^1 \equiv \tilde{y} \ , \ x^i\}$ which has the form (1) with

$$\tilde{G}_{11} = M^{-1} \ \ , \ \ \tilde{G}_{1i} = M^{-1}B_{1i} \ \ , \ \ \tilde{G}_{ij} = G_{ij} - M^{-1}(G_{1i}G_{1j} - B_{1i}B_{1j}) \ \ ,$$

$$\tilde{B}_{1i} = M^{-1}G_{1i} \ \ , \ \ \tilde{B}_{ij} = B_{ij} - M^{-1}(B_{1i}G_{1j} - G_{1i}B_{1j}) \ \ , \quad M \equiv G_{11} \ \ . \tag{2}$$

The integral over p_a is, however, ill-defined given that the coefficient $M(x)$ of the p^2 term is just a local function and not an elliptic differential operator. Therefore one cannot assert that the partition functions corresponding to action functionals I and $\tilde{I}$ are equal. However one may expect that the ratio of these partition functions should be given by the exponential of a local functional

$$\Delta I = \frac{1}{4\pi} \int d^2z\sqrt{g}\ [c_0 \mathrm{e}^{2\lambda} + c_1\partial_a\lambda\partial^a\lambda \ + \ c_2 R\lambda\ ] \ \ , \quad M \equiv \mathrm{e}^{2\lambda} \ \ . \tag{3}$$

[1] For duality transformations in the non-abelian case see e.g.[5].

The coefficients c_n should depend on a particular way of defining the integral over p_a. The definition suggested in [1] was based on changing the variables

$$p_a = \partial_a \xi^1 + \epsilon_a{}^b \partial_b \xi^2 \quad , \tag{4}$$

and computing the determinant of the corresponding second order differential operator Q defined on the scalar fields ξ^n. The part of this determinant which depends on the conformal factor of the $2d$ metric g_{ab} is determined by the Weyl anomaly and can be easily extracted [1,3]. As a result, one finds that c_2 in (3) is equal to –1, implying that the dilaton should be shifted under the duality by $-\frac{1}{2}\log M$. There remains, however, the question about other possible terms in the determinant of Q which are Weyl invariant but M - dependent. It is not clear why such terms in det Q should be local. This question was not resolved in [1,3].[2] We shall present the exact computation of log det Q in Appendix and show that it is given by a local expression (3) (with non-vanishing c_1). The $\partial_a\lambda\partial^a\lambda$ term coming from log det Q is cancelled by the contribution of another determinant (ignored in [1]) which appears in the process of duality transformation in the path integral.

In general, one would like to fix the structure of finite local counterterms (3) in such a way that the original and dual sigma models remain equivalent at the quantum level. The necessary condition for the equivalence is that (3) should compensate for the non-trivial ratio of the determinants which appear as a result of integrating over y and $\tilde{y}$ in the original and dual theories,

$$Z = \frac{\int [dy] \exp[-\int d^2z \sqrt{g} M(x) \partial_a y \partial^a y]}{\int [d\tilde{y}] \exp[-\int d^2z \sqrt{g} M^{-1}(x) \partial_a \tilde{y} \partial^a \tilde{y}]} \quad , \tag{5}$$

Ignoring the zero mode factors

$$Z = [\det\, \Delta_0]^{-1/2} [\det\, \tilde{\Delta}_0]^{1/2} \quad . \tag{6}$$

However, since each of the determinants in (6) is a complicated non-local functional of the function M it is not *a priori* clear why the logarithm of their ratio (6) should have a local

[2] It was noted in [3] that det Q may be exactly computable using the standard variational argument but a consistent derivation was not carried out.

form. In Sec.3 below we shall prove that (6) is, in fact, given by a local expression which structure in general depends on choices of the scalar products in the corresponding spaces of y and $\tilde{y}$. For the natural choice of scalar products we will find that

$$Z = \exp(\frac{1}{8\pi}\int d^2z\sqrt{g}\ R\log M) \quad , \tag{7}$$

implying that the dilaton is shifted under the duality transformation [1]

$$\tilde{\phi} = \phi - \frac{1}{2}\log M \quad . \tag{2'}$$

The proof will be based on the interpretation of the partition function (5),(6) as the torsion of the elliptic complex of exterior differentials acting on forms (the DeRham complex) in $d = 2$ in the case of non-trivial scalar products in the corresponding vector spaces. The result may be considered as a generalisation of the standard expression which gives the index of the DeRham complex in terms of the Euler number (see e.g. [7]).

2. Basic definitions and relations

We shall start with summarising some relevant information (following ref.[8]) about a special combination of determinants of elliptic operators which is known as the torsion of an elliptic complex. Let us consider a complex (E_i, T_i) $(i = 0, 1, ..., N\ , E_{-1} = E_{N+1} = 0)$, i.e. a sequence of vector spaces E_i and linear operators T_i acting from the space E_i to the space E_{i+1} and satisfying $T_{i+1}T_i = 0$. Let $< \ , \ >_i$ denote a scalar product in E_i and define the adjoint operators $T_i^* : E_{i+1} \to E_i$ by $< a, T_i b >_{i+1} = < T_i^* a, b >_i$. If the self-adjoint operators $\Delta_i : E_i \to E_i$

$$\Delta_i = T_i^* T_i + T_{i-1}T_{i-1}^* \tag{8}$$

are elliptic differential operators the complex (E_i, T_i) is called an elliptic complex. The torsion $Z(E_i, T_i)$ of the elliptic complex (E_i, T_i) is given by the formula[3]

$$\log Z(E_i, T_i) = \frac{1}{2}\sum_{i=0}^{N}(-1)^i(i+1)\log\det\Delta_i \quad , \tag{9}$$

[3] Note that we have changed the notation as compared to ref.[8]: our T_i is T_{N-i} of [8].

where the regularised determinants of operators Δ_i are defined by means of ζ-function. Z can be identified with the partition function of a degenerate functional (associated with $(E_i,\ T_i)$) with zero action [8].

Using that

$$T_{i+1}T_i = 0 \quad , \quad \det\ \Delta_i = \det\ (T_i^*T_i)\ \det\ (T_{i-1}T_{i-1}^*) \quad , \quad \det\ (T_iT_i^*) = \det\ (T_i^*T_i) \quad , \tag{10}$$

one can represent Z in the form

$$Z = \prod_{i=0}^{N} (\det\ T_i^* T_i)^{\frac{1}{2}(-1)^{i+1}} \quad . \tag{11}$$

Since T_i^* depend on a definition of the scalar products in E_i , Z changes under a variation of the scalar products. If one redefines the scalar products by inserting the operators $M_i : E_i \to E_i$,

$$< \ , \ >_i' = < M_i \ , \ >_i \quad , \qquad T_i^{*\prime} = M_i^{-1} T_i^* M_{i+1} \quad . \tag{12}$$

As was shown in [8], the variation of the partition function (9),(11) under the variation of the scalar products in E_i can be expressed in terms of the Seeley coefficients (i.e. local functionals appearing in the $t \to 0$ asymptotics of $\exp(-t\Delta_i)$) and the zero modes of Δ_i. If

$$\delta\lambda_i = \frac{1}{2} M_i^{-1} \delta M_i$$

are the operators describing an infinitesimal change of the scalar products,[4]

$$\delta\ \log\ Z = \sum_{i=0}^{N} (-1)^i [\Psi_0(\delta\lambda_i|\Delta_i) - P(\delta\lambda_i|\Delta_i)] \quad . \tag{13}$$

Here Ψ_0 is the coefficient of t^0 in the asymptotic expansion

$$\mathrm{Tr}(\delta\lambda_i \mathrm{e}^{-t\Delta}) \sim \sum_k \Psi_k(\delta\lambda_i|\Delta_i)\ t^k \quad , \tag{14}$$

[4] The proof of this formula is based on representing the determinants in terms of proper-time integrals of the heat kernels and relating the variations of heat kernels corresponding to different operators.

and

$$P(\delta\lambda_i|\Delta_i) = \sum_n < \delta\lambda_i f_i^{(n)}, f_i^{(n)} >_i \quad , \tag{15}$$

where $f_i^{(n)}$ is a basis in the kernel of Δ_i.

Let us now specialise to the case when (E_i, T_i) is the DeRham complex, i.e. when E_i are the spaces of i-forms on a compact riemannian space $\mathcal{M}^N$ with the metric g_{ab} and $T_i = d_i$ are exterior differentials. The torsion (9),(11) can be considered as the partition function of the quantum theory of the antisymmetric tensor of rank N which has zero action [8–11]. We are interested in the generalisation of the standard discussion to the case when the scalar products in E_i are non-trivial, namely contain additional scalar functions $M_i(z)$

$$< \omega, \sigma >_i = \int d^N z \sqrt{g}\ M_i g^{a_1 b_1} ... g^{a_i b_i}\ \omega_{a_1 ... a_i} \sigma_{b_1 ... b_i} \quad . \tag{16}$$

Then according to (12), (8)

$$T_i^* = M_i^{-1} d_i^* M_{i+1} \quad , \tag{17}$$

$$\Delta_i = M_i^{-1} d_i^* M_{i+1} d_i + d_{i-1} M_{i-1}^{-1} d_{i-1}^* M_i \quad . \tag{18}$$

When $M_i = 1$ the partition function Z (11) is known as the Ray-Singer torsion $tor\mathcal{M}$ of the manifold $\mathcal{M}^N$ [9,11] the variation of which under a change of the metric g_{ab} can be expressed in terms of the "anomalies" and zero modes of Δ_i ($tor\mathcal{M}$ is equal to 1 if N is even). Note that $\det \Delta_i \neq \det \Delta_{N-i}$ when $M_i \neq 1$.

The variation of Z (13) under a variation of M_i takes the form

$$\delta \log Z = \sum_{i=0}^{N} (-1)^i \ [\int d^N z\ \delta\lambda_i\ b_N(z|\Delta_i) - \sum_n < \delta\lambda_i f_i^{(n)}, f_i^{(n)} >_i] \quad , \tag{19}$$

$$M_i \equiv \mathrm{e}^{2\lambda_i} \quad .$$

Here $b_N(z|\Delta_i)$ is the local Seeley coefficient (integrand of Ψ_0) of the Laplacian Δ_i and $f_i^{(n)}$ are the zero modes of Δ_i (for a discussion of the zero mode contribution see [11]). Note that for $M_i = 1$

$$\sum_{i=0}^{N} (-1)^i \int d^N z\ b_N(z|\Delta_i)$$

is the index of the DeRham complex which by index theorem is equal to the Euler number χ of $\mathcal{M}^N$. Eq.(19) then implies that (up to the contribution of the zero modes) the power of the common constant scale of M_i in Z is equal to $\frac{1}{2}\chi$. If $b_N(z|\Delta_i)$ and $f_i^{(n)}$ are known explicitly, eq.(19) gives a system of functional equations

$$\frac{\delta \log Z}{\delta\lambda_i} = F(\lambda_1, ..., \lambda_N; g_{ab})$$

which, in principle, can be integrated to find the dependence of Z on M_i.

3. Torsion of generalised DeRham complex in two dimensions

The case of our interest is when the complex is defined on a two-dimensional Riemann space, i.e. the simplest non-trivial case of the above construction ($N = 2$),

$$0 \to E_0 \to E_1 \to E_2 \to 0 \quad .$$

d_0 acts from the space of scalars E_0 to the space of vectors (1-forms) E_1 and d_1 acts from E_1 to the space of 2-forms E_2. In two dimensions 2-forms can be identified with ("dual") scalars ($\omega_{ab} = \epsilon_{ab}\omega$). The corresponding scalar products and Laplacians are given by (16),(18)

$$< \omega, \sigma >_0 = \int d^2z\sqrt{g}\ M_0\ \omega\sigma \quad , \quad < \omega, \sigma >_1 = \int d^2z\sqrt{g}\ M_1\ g^{ab}\omega_a\sigma_b \quad ,$$

$$< \omega, \sigma >_2 = \int d^2z\sqrt{g}\ M_2\ g^{ab}g^{cd}\ \omega_{ac}\sigma_{bd} \quad , \quad , \tag{20}$$

$$\Delta_0 = M_0^{-1}d_0^*M_1d_0 \quad , \quad \Delta_1 = M_1^{-1}d_1^*M_2d_1 + d_0M_0^{-1}d_0^*M_1 \quad , \quad \Delta_2 = d_1M_1^{-1}d_1^*M_2 \quad ,$$

$$\Delta_0 = -M_0^{-1}\nabla^a(M_1\nabla_a) \quad , \quad \Delta_2 = -\nabla^a(M_1^{-1}\nabla_aM_2)$$

$$\Delta_{1ab} = -M_1^{-1}\nabla^c(M_2(g_{ab}\nabla_c - g_{cb}\nabla_a)) - \nabla_a(M_0^{-1}\nabla_bM_1) \quad . \tag{21}$$

Since the determinants are invariant under $\Delta \to S^{-1}\Delta S$ where S is an operator of multiplication by a function Δ_2 can be represented also in the equivalent form

$$\Delta_2 = -M_2\nabla^a(M_1^{-1}\nabla_a) \quad . \tag{21'}$$

The partition function Z in (9),(11) is given by

$$Z = [\det\ \Delta_0]^{1/2}[\det\ \Delta_1]^{-1}[\det\ \Delta_2]^{3/2} \tag{22}$$

$$= [\det\ \Delta_2]^{-1/2}\det\ \Delta_1[\det\ \Delta_0]^{-3/2}$$

or

$$Z = [\det\ \Delta_0]^{-1/2}[\det\ \Delta_2]^{1/2} \quad . \tag{23}$$

Comparing (22),(23) with (5),(6) we conclude that Z in (23) gives the well-defined expression for the ratio of the partition functions of the two dual scalar theories in the general case of arbitrary functions in the scalar products ($M = M_1$). Our aim will be to compute Z explicitly as a functional of $M_i(z)$ and g_{ab} using the variational relation (19). Let us first ignore the contribution of the zero modes. Then (19) takes the form

$$\delta\ \log\ Z = \int d^2z\ [\ \delta\lambda_0 b_2(z|\Delta_0) - \delta\lambda_1 b_2(z|\Delta_1) + \delta\lambda_2 b_2(z|\Delta_2)\ ] \quad . \tag{24}$$

The variation of Z under the Weyl rescaling of the metric $\delta g_{ab} = 2\delta\rho g_{ab}$ can be represented as a particular case of (24) with $\delta\lambda_0 = \delta\rho\ ,\quad \delta\lambda_1 = 0\ ,\quad \delta\lambda_2 = -\delta\rho$, i.e.

$$\delta\ \log\ Z = \int d^2z\ \delta\rho\ [\ b_2(z|\Delta_0) - b_2(z|\Delta_2)\ ] \quad . \tag{25}$$

This expression, being the difference of the conformal anomalies of the scalar theory and its dual, is also an obvious consequence of (23).

We shall use the following standard result for the Seeley coefficients of the Laplace-type operators (21) in two dimensions [12]. Consider the elliptic differential operator

$$\Delta = -Ig^{ab}\partial_a\partial_b - 2A^a\partial_a + Y \quad , \tag{26}$$

where $a, b = 1, 2$, $g^{ab}(z)$ is positive definite, I is $n \times n$ identity matrix and A^a and Y are $n \times n$ matrix valued functions in R^2. Then

$$b_2(z|\Delta) = \frac{1}{4\pi}\sqrt{g}\mathrm{Tr}(\frac{1}{6}RI - \nabla_a A^a - g_{ab}A^a A^b - Y) \quad , \tag{27}$$

where g_{ab} is the inverse of g^{ab}, R is the curvature of g_{ab} and $\nabla_a A^a = \frac{1}{\sqrt{g}}\partial_a(\sqrt{g}A^a)$. The operator (26) can be represented also as

$$\Delta = -g^{ab} D_a D_b + X \quad , \tag{28}$$

where the covariant derivative D_a contains both the Christoffel and A_a connection terms. The equivalent form of (27) is[5]

$$b_2(z|\Delta) = \frac{1}{4\pi}\sqrt{g}\ \mathrm{Tr}\ (\frac{1}{6}IR - X) \quad . \tag{29}$$

The operators Δ_0 and Δ_2 in (21) can be represented in the form (26) (with the metric rescaled by M_1/M_0 and M_2/M_1). Applying (27) and returning back to the original metric in (21) we find

$$b_2(z|\Delta_0) = \frac{1}{4\pi}\sqrt{g}\ [\frac{1}{6}R + \frac{1}{3}\nabla^2(\lambda_1 - \lambda_0) - \nabla^2\lambda_1 - \partial_a\lambda_1\partial^a\lambda_1] \quad , \quad M_i \equiv \mathrm{e}^{2\lambda_i} \quad , \tag{30}$$

$$b_2(z|\Delta_2) = \frac{1}{4\pi}\sqrt{g}\ [\frac{1}{6}R + \frac{1}{3}\nabla^2(\lambda_2 - \lambda_1) + \nabla^2\lambda_1 - \partial_a\lambda_1\partial^a\lambda_1] \quad . \tag{31}$$

Substituting (30),(31) into (25) and integrating the resulting equation we find the dependence of Z on the metric g_{ab}[6]

$$\log\ Z = \frac{1}{4\pi}\int d^2z\sqrt{g}\ [R\lambda_1 + \frac{1}{6}R(\lambda_0 + \lambda_2 - 2\lambda_1)] + O(\lambda_0, \lambda_1, \lambda_2) \quad . \tag{32}$$

For the case of equal M_i ($\lambda_i = \lambda$) this expression was found in [3]. To determine the λ_i-dependent terms in (32) we need to integrate eq.(24). Since for general M_i the vector operator Δ_1 in (21) is not of the type (26) we shall do this in two steps. First, we substitute (30) and (31) into (24) and integrate over λ_0 and λ_2. Taking into account (32) we get

$$\log\ Z = \frac{1}{4\pi}\int d^2z\sqrt{g}\ [\ R\lambda_1 + \frac{1}{6}R(\lambda_0 + \lambda_2 - 2\lambda_1)$$

[5] Note that b_2 is invariant under the similarity transformation $\Delta \to S^{-1}\Delta S$, where S is the operator of multiplication by function.

[6] We consider the case of the spherical topology so the dependence on the metric is determined by the dependence on its conformal factor.

$$+\frac{1}{6}(\lambda_0-\lambda_2)\nabla^2(2\lambda_1-\lambda_0-\lambda_2)-(\lambda_0-\lambda_2)\nabla^2\lambda_1-(\lambda_0+\lambda_2)\partial_a\lambda_1\partial^a\lambda_1)]+O(\lambda_1) \quad . \tag{33}$$

To determine the terms which depend only on λ_1 we consider the special case when all λ_i are equal to λ. Then Δ_1 in (21) takes the form

$$\Delta_{1ab}=-\mathrm{e}^{-2\lambda}\nabla^c\mathrm{e}^{2\lambda}(g_{ab}\nabla_c-g_{cb}\nabla_a)-\nabla_a\mathrm{e}^{-2\lambda}\nabla_b\mathrm{e}^{2\lambda}$$

$$=-g_{ab}\nabla^2+R_{ab}-2g_{ab}\partial^c\lambda\nabla_c-2\nabla_a\nabla_b\lambda \quad . \tag{34}$$

Now (26)–(29) can be applied and one finds

$$b_2(z|\Delta_1)|_{\lambda_i=\lambda}=\frac{1}{4\pi}\sqrt{g}\ (\frac{1}{3}R-R-2\partial_a\lambda\partial^a\lambda) \quad . \tag{35}$$

Substituting (30),(31) and (35) into (24) and assuming $\delta\lambda_i=\delta\lambda$ we get

$$\log\ Z|_{\lambda_i=\lambda}=\frac{1}{4\pi}\int d^2z\sqrt{g}\ R\lambda \quad . \tag{36}$$

The origin of the "anomalous" term (36) can be attributed to the presence of the R_{ab} term in the vector operator (34) (which gives the $-R$ contribution to (35)). Comparing (33) to (36) we determine that the $O(\lambda_1)$ term in (33) must be $2\lambda_1\partial_a\lambda_1\partial^a\lambda_1$, i.e the final expression for Z (up to zero mode factors) is

$$\log\ Z=\frac{1}{4\pi}\int d^2z\sqrt{g}\ [\ R\lambda_1+\frac{1}{6}R(\lambda_0+\lambda_2-2\lambda_1)$$

$$+\frac{1}{6}(\lambda_0-\lambda_2)\nabla^2(2\lambda_1-\lambda_0-\lambda_2)-(\lambda_0-\lambda_2)\nabla^2\lambda_1-(\lambda_0+\lambda_2-2\lambda_1)\partial_a\lambda_1\partial^a\lambda_1)] \quad . \tag{37}$$

It is easy to check that the derivative of (37) with respect to λ_1 is equal to (35) for $\lambda_i=\lambda$. Note that in agreement with the index theorem the common constant scale of M_i appears in log Z with the coefficient equal to one half of the Euler number.

Let us now include the contribution of the zero modes. We shall assume that Δ_1 does not have zero modes. The normalised zero modes of Δ_0 and Δ_2 are

$$f_0=(\int d^2z\sqrt{g}\ M_0)^{-1/2} \quad , \quad f_2=M_2^{-1}(\int d^2z\sqrt{g}\ M_2^{-1})^{-1/2} \quad . \tag{38}$$

Including the corresponding zero mode terms (see (19)) into (24) and integrating over λ_0 and λ_2 we get the following additional terms in Z (23)

$$(\log Z)_{z.m.} = -\frac{1}{2}\log\,(\int d^2z\sqrt{g}\ M_0) + \frac{1}{2}\log\,(\int d^2z\sqrt{g}\ M_2^{-1}) \quad . \tag{39}$$

As a result, we can represent (23) in the form

$$Z = Z'\ (\int d^2z\sqrt{g}\ M_0)^{-1/2}\ (\int d^2z\sqrt{g}\ M_2^{-1})^{1/2} \quad , \tag{40}$$

where Z' is the local expression given by (37).

The case of equal functions M_i is the one which is relevant for the duality transformation problem discussed at the beginning of the paper. In fact, given the sigma model (1) it is natural to define the scalar product in the tangent space at x^μ as

$$< \delta x, \delta x' >= \int d^2z\sqrt{g}G_{\mu\nu}(x)\delta x^\mu \delta x'^\nu = \int d^2z\sqrt{g}M(x)\delta y\delta y' + ... \quad . \tag{41}$$

The scalar product corresponding to the dual sigma model (2) is then

$$< \delta\tilde{x}, \delta\tilde{x}' >= \int d^2z\sqrt{g}\tilde{G}_{\mu\nu}(x)\delta\tilde{x}^\mu \delta\tilde{x}'^\nu = \int d^2z\sqrt{g}M^{-1}(x)\delta\tilde{y}\delta\tilde{y}' + ... \quad . \tag{42}$$

The scalar product on scalars δy is thus determined by the function $M_0 = M$. As it is clear from the structure of the quadratic functionals (actions) in (5), to be able to identify M with the scalar product function in the space of vectors we need to rescale $\tilde{y}$ by a factor of M, i.e. we should identify $M^{-1}\tilde{y}$ with the 2-forms which appeared in the above discussion (so that $\tilde{y}$ = const corresponds to the zero mode, etc). Then the scalar product on the 2-forms is also defined by M (i.e. all M_i are equal to M) and the operators Δ and $\tilde{D}$ in (6) are equivalent to Δ_0 and Δ_2. For $M_i = M$ eq.(40) takes the form (see (36))

$$\begin{aligned}&(\int d^2z\sqrt{g}M)^{1/2}\ [\det\ \Delta_0]^{-1/2}\\ &= \exp(\frac{1}{8\pi}\int d^2z\sqrt{g}\ R\log M)\ (\int d^2z\sqrt{g}M^{-1})^{1/2}\ [\det\ \Delta_2]^{-1/2} \quad .\end{aligned} \tag{43}$$

Note that if M = const it drops out from (43) (we have assumed that the 2-space has the topology of a sphere). As a result, we have proved that up to the ratio of the zero mode factors

$$\frac{\int dy(\int d^2z\sqrt{g}M)^{1/2}}{\int d\tilde{y}(\int d^2z\sqrt{g}M^{-1})^{1/2}} \tag{44}$$

(5) is indeed given by the local expression (7).

4. Concluding remarks

In conclusion, let us make several comments.

Eqs.(5),(7),(43) have a straightforward generalisation to the case when the sigma model (1) has a number of commuting isometries, i.e. when y and $\tilde{y}$ have an additional index s. Then $G_{st} \equiv M_{st}$ is a matrix depending on the rest of the fields $x^i(z)$ and M in (7),(43) should be replaced by det M.

The relation (43) proved above may be useful in trying to clarify the issue of modification of the leading order duality transformations (2),(2′) by higher loop effects [3].

Equations (33) and (43) may be of interest also from a mathematical point of view giving the explicit dependence of the torsion of the $N = 2$ DeRham complex on the scalar products and providing the "$M \neq$ const" generalisation of the corresponding index theorem. It may be of interest to study higher dimensional analogs of (33),(43) (in particular, in connection with possible higher dimensional analogs of sigma model duality).

The torsion (22),(23) can be interpreted as the partition function of the theory of rank 2 antisymmetric tensor in two dimensions [8,11] (i.e. of a 'topological' theory with zero action). What we have found is that if non-constant functions are included in the definition of the scalar products the resulting "effective action" (36),(37) is a *local* action of $2d$ gravity coupled to scalar(s). Thus the $2d$ $scalar - tensor$ gravity appears as an 'induced' theory.

The methods of [8] and of the present paper may find applications in other contexts (e.g. in gauged WZW theory) where a careful account of the dependence on the definition of the scalar products is important. The resulting ambiguity in a choice of local counterterms should be fixed by additional conditions depending on particular theory.

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Appendix . Integration over "non-dynamical" vector field and determinant of operator Q

1. Let us consider the following integral over the $2d$ vector field p_a

$$Z_1 = \int [dp_a] \exp [-\frac{1}{2}\int d^2z\sqrt{g}\ M_1(z) g^{ab} p_a p_b\ ] \quad , \tag{A.1}$$

where M_1 is a given function. We are assuming that the measure (scalar product) for p_a is trivial (if M_1 is present also in the scalar product it is natural to set $Z_1 = 1$). This integral is not well defined. Using different definitions (regularisations) of (A.1) one will get expressions which will differ by local (finite) counterterms. A choice of particular definition is dictated by some additional conditions which the total theory (where (A.1) appears at some intermediate step) should satisfy.

Given that $\Delta_1 = d_1^* d_1 + d_0 d_0^* = (-g_{ab}\nabla^2 + R_{ab})$ is a Laplace operator on vectors one may define (A.1), for example, as (cf.(34),(35))

$$Z_1 = \exp[-\frac{1}{2}\mathrm{Tr}\ (\log\ M_1\ \mathrm{e}^{-\epsilon\Delta_1})]$$

$$= \exp[-\frac{1}{2}\int d^2z\sqrt{g}\ \log\ M_1(\frac{2}{\epsilon} + \frac{1}{3}R - R)] \quad . \tag{A.2}$$

It is more natural, however, to use the following general idea: if M is a "bad" operator one can define its determinant by introducing an auxiliary operator P such that P^*MP and P^*P are "good" operators and setting

$$\det\ M \equiv \frac{\det\ P^*MP}{\det\ P^*P} \quad . \tag{A.3}$$

The result will, in general, *depend* on a choice of P. However the dependence of P can be calculated; see [8]. Changing the variables from p_a to a pair of scalar fields ξ^n

$$p_a = \partial_a\xi^1 + \epsilon_a{}^b\partial_b\xi^2 \equiv P_{an}\xi^n \quad , \tag{A.4}$$

one can represent (A.1) as

$$Z_1 = \ J\ Z_Q \quad , \quad Z_Q \equiv [\det\ Q]^{-1/2} \quad , \quad J = [Z_Q(M_1 = 1)]^{-1} \quad , \tag{A.5}$$

where J is the Jacobian of the transformation and Q is the following Laplacian defined on a pair of scalars

$$Q \equiv P^\dagger P \quad , \qquad P^\dagger \equiv M_0^{-1} P^* M_1 \quad , \tag{A.6}$$

$$Q_{nm} = -M_0^{-1} \delta_{nm} \nabla^a (M_1 \partial_a) - \epsilon_{nm} \epsilon^{ab} M_0^{-1} \partial_a M_1 \partial_b$$

$$= \mathrm{e}^{2(\lambda_1 - \lambda_0)} [\, -\delta_{nm} \nabla^2 - 2(\delta_{nm} g^{ab} + \epsilon_{nm} \epsilon^{ab}) \partial_a \lambda_1 \partial_b \,] \quad , \tag{A.7}$$

$$M_i \equiv \mathrm{e}^{2\lambda_i} \quad .$$

Here

$$P^{*na} p_a = (-\nabla^a p_a \ , \quad -\epsilon^{ab} \nabla_a p_b) \quad , \tag{A.8}$$

and we have assumed that the scalar product in the space of ξ^n is defined with an extra function $M_0(z)$ (cf.(20)).

Since Q is a $2d$ scalar operator the dependence of Z_Q on the conformal factor ρ of the 2-metric is determined by the (Weyl) anomaly. In view of the "product" structure of Q the same is effectively true also for the dependence on M_0 and M_1. Computing the variations of det Q with respect to M_i and ρ we get (cf.(24),(25))

$$\delta \log Z_Q = \int d^2 z \ [\ (\delta\rho + \delta\lambda_0) b_2(z|Q) - \delta\lambda_1 b_2(z|\tilde{Q})\ ] \quad . \tag{A.9}$$

Here the operator $\tilde{Q} = P P^\dagger$ or, equivalently,

$$\tilde{Q} = M_1 P M_0^{-1} P^* \quad , \tag{A.10}$$

is a Laplacian acting on vectors,

$$\tilde{Q}_{ab} = -M_1 \nabla^c M_0^{-1} (g_{ab} \nabla_c - g_{cb} \nabla_a) - M_1 \nabla_a M_0^{-1} \nabla_b$$

$$= \mathrm{e}^{2(\lambda_1 - \lambda_0)} [\, -g_{ab} \nabla^2 + R_{ab} + 2(g_{ab} g^{cd} + \epsilon_{ab} \epsilon^{cd}) \partial_c \lambda_0 \nabla_d \,] \quad . \tag{A.11}$$

Note that det $\tilde{Q}$ = det Q and that the structure of the operator $\tilde{Q}$ is different from the structure of Δ_1 in (21). The Seeley coefficients $b_2(z|Q)$ and $b_2(z|\tilde{Q})$ are found using (26),(27) (cf.(30),(31),(34))

$$b_2(z|Q) = \frac{1}{4\pi}\sqrt{g}\ [\frac{1}{3}R + \frac{2}{3}\nabla^2(\lambda_1 - \lambda_0) - 2\nabla^2\lambda_1\ ]\ \ , \qquad (A.12)$$

$$b_2(z|\tilde{Q}) = \frac{1}{4\pi}\sqrt{g}\ [\frac{1}{3}R - R + \frac{2}{3}\nabla^2(\lambda_1 - \lambda_0) + 2\nabla^2\lambda_0\ ]\ \ . \qquad (A.13)$$

Integrating (A.9) we get (cf.(37))

$$\log\ Z_Q(\lambda_0, \lambda_1) = \frac{1}{4\pi}\int d^2z\sqrt{g}\ [\ -\frac{1}{12}R\nabla^{-2}R + R\lambda_1$$

$$-\frac{1}{3}R(\lambda_1 - \lambda_0) - \frac{1}{3}(\lambda_0 - \lambda_1)\nabla^2(\lambda_0 - \lambda_1) - 2\lambda_0\nabla^2\lambda_1\ ]\ \ , \qquad (A.14)$$

$$\log\ Z_1 = \log\ Z_Q(\lambda_0, \lambda_1) - \log\ Z_Q(\lambda_0, 0)$$

$$= \frac{1}{4\pi}\int d^2z\sqrt{g}\ [\ R\lambda_1 - \frac{1}{3}R\lambda_1 + \frac{1}{3}\lambda_1\nabla^2(2\lambda_0 - \lambda_1) - 2\lambda_0\nabla^2\lambda_1\ ]\ \ . \qquad (A.15)$$

In the special case when $M_0 = M_1 = M$ (i.e. $\lambda_0 = \lambda_1 = \lambda$) which is relevant for the discussion of the duality transformation in the path integral one has

$$Q_{nm} = -\delta_{nm}\nabla^2 - 2(\delta_{nm}g^{ab} + \epsilon_{nm}\epsilon^{ab})\partial_a\lambda\partial_b\ \ , \qquad (A.16)$$

$$\tilde{Q}_{ab} = -g_{ab}\nabla^2 + R_{ab} + 2(g_{ab}g^{cd} + \epsilon_{ab}\epsilon^{cd})\partial_c\lambda\nabla_d\ \ , \qquad (A.17)$$

$$\log\ Z_Q|_{\lambda_i=\lambda}\ = \frac{1}{4\pi}\int d^2z\sqrt{g}\ [\ -\frac{1}{12}R\nabla^{-2}R + R\lambda + 2\partial_a\lambda\partial^a\lambda\ ]\ \ . \qquad (A.18)$$

It is possible to check the presence of the $\partial_a\lambda\partial^a\lambda$ term in log Z_Q by an explicit perturbative calculation. Changing the variables from ξ^n in (A.4) to

$$u^n = \mathrm{e}^{-\lambda}\xi^n$$

for which the scalar product will be λ-independent one can represent Z_Q as a path integral with the following action (cf.(A.16))

$$I = \frac{1}{2}\int d^2z\sqrt{g}\ [\partial^a u^n\partial_a u_n\ + 2\epsilon_{nm}\epsilon^{ab}\partial_b\lambda u^n\partial_a u^m + (\nabla^2\lambda + \partial_a\lambda\partial^a\lambda)u^n u_n]\ \ ,$$

or

$$I = \frac{1}{2} \int d^2 z \sqrt{g} \ (D^a u^n D u_n \ - F u^n u_n) \ ,$$

$$D_a u^n = \partial_a u^n + A^n_{am} u^m \quad , \quad A^n_{am} = \epsilon^n{}_m \epsilon_a{}^b \partial_b \lambda \ , \quad F \equiv \frac{1}{4} \epsilon^{ab} \epsilon_{nm} F^{nm}_{ab} = -\nabla^2 \lambda \ .$$

Computing the $O(A^2)$ term in the effective action on a flat background using dimensional regularisation[7] one finds the finite Schwinger-type term

$$\log \ Z_Q = \frac{1}{4\pi} \int d^2 z [-\text{Tr} \ (A^{\perp}_a)^2 + ...]$$

($A^{\perp}_a$ is the transverse part of A_a). This result is in agreement with (A.18).

2. The operator equivalent to Q with $M_0 = M_1 = M$ was considered in [1,3] and the presence of the term $\frac{1}{8\pi} \int d^2 z \sqrt{g} \ R \log \ M$ in the logarithm of its determinant was established by computing the conformal anomaly. We have found that the complete expression for log det Q (A.18) contains also the *additional* M-dependent term

$$\frac{1}{8\pi} \int d^2 z \sqrt{g} \ \partial_a \log \ M \partial^a \log \ M \ . \tag{A.19}$$

By formally doing the duality transformation in the path integral one could expect [1,3] that the ratio (5),(6) should be equal to Z_1 (A.1) or to Z_Q. Comparing (37),(43) with (A.14),(A.15),(A.18) we conclude that in fact this cannot be true since the term (A.19) in (A.18) cannot be present in the logarithm of (5) (the latter must change sign under $M \to M^{-1}$).

This apparent contradiction is resolved by discovering that the actual relation between the torsion (5),(23) and Z_Q contains also a contribution of another determinant of second

[7] It is easy to check that UV and IR divergences cancel separately so that one may set the massless tadpole equal to zero and use the standard integrals

$$\int \frac{d^d p \ p_a p_b}{(2\pi)^d p^2 (p-q)^2} = -\frac{1}{4(d-1)} (q^2 \delta_{ab} - d q_a q_b) J \ , \quad J = \int \frac{d^d p}{(2\pi)^d p^2 (p-q)^2} = \frac{1}{\pi(d-2)} + ... \ ,$$

as well as $\epsilon^{ac} \epsilon_{ad} = \delta^c_d$.

order elliptic operator. To derive the exact form of this relation let us repeat the standard steps corresponding to the duality transformation at the path integral level.[8] Let us start with the following path integral

$$\bar{Z} = \int [dp_a][d\tilde{y}] \exp\,(- \int d^2z\sqrt{g}\ [\ \frac{1}{2}M(z)g^{ab}p_a p_b + i\epsilon^{ab}p_a\partial_b\tilde{y}]) \quad . \tag{A.20}$$

Integrating over p_a we get the path integral of the dual theory $\bar{Z} \sim [\det\,\tilde{\Delta}_0]^{-1/2}$ (cf.(5),(6)). If one integrates first over $\tilde{y}$ one obtains the δ-function implying that $p^a = \partial_a y$, i.e. the path integral becomes that of the original theory, i.e. $\bar{Z} \sim [\det\,\Delta_0]^{-1/2}$. There is, however, an additional determinant which appears from the δ-function. To give a precise sense to the above relations let us first change the variables as in (A.4)

$$p_a = \partial_a y + \epsilon_a{}^b\partial_b\bar{y} \quad , \tag{A.21}$$

$$\bar{Z} = J\int [dy][d\bar{y}][d\tilde{y}] \exp\,(- \int d^2z\sqrt{g}\ [\ \frac{1}{2}M(z)(\partial_a y + \epsilon_a{}^b\partial_b\bar{y})^2 + i\partial^a\bar{y}\partial_a\tilde{y}\ ]) \quad . \tag{A.22}$$

Here J is the Jacobian corresponding to (A.21) (see (A.5)). In the context of the sigma model duality transformation we should assume that the scalar products in the spaces of y and $\bar{y}$ are defined with the function $M_0 = M$ while the scalar product in the space of $\tilde{y}$ contains M^{-1} (cf.(41),(42)). Integrating first over $\bar{y}$ and y and then over $\tilde{y}$ we get (cf.(A.5))[9]

$$\bar{Z} = \ J\ [\det\,Q]^{-1/2}\ [\det\,\tilde{\Delta}_0]^{-1/2} \quad . \tag{A.23}$$

Integrating first over $\tilde{y}$ we get the δ-function factor which can be represented as $\delta(\bar{y})$ multiplied by

$$Z_H = \int [d\bar{y}][d\tilde{y}] \exp\,(-i \int d^2z\sqrt{g}\ \partial^a\bar{y}\partial_a\tilde{y}\ ) \equiv [\det\,H]^{-1/2} \quad . \tag{A.24}$$

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[8] It should be noted that the discussion which follows is based on operations with ill-defined integrals and thus is not rigorous.

[9] To compute the integral over $\bar{y}$ and y one is to make a (non-local) shift of the fields which can be determined from the corresponding shift of p_a in (A.20).

Then

$$\bar{Z} = \ J \ [\det \ H]^{-1/2} \ [\det \ \Delta_0]^{-1/2} \quad . \tag{A.25}$$

Comparing (A.23) with (A.25) we find that the following relation must be true

$$\det \ Q = \det \ H \ \det \ \Delta_0 \ [\det \ \tilde{\Delta}_0]^{-1} \quad , \tag{A.26}$$

i.e.

$$Z = \frac{Z_Q}{Z_H} \quad . \tag{A.27}$$

Z and Z_Q defined in (6),(23) and (A.5) (with $M_i = M$) were already computed in (36) and (A.18). According to (A.27)

$$\log \ Z_H = \frac{1}{4\pi} \int d^2 z \sqrt{g} \ [\ -\frac{1}{12} R \nabla^{-2} R + 2 \partial_a \lambda \partial^a \lambda \ ] \quad . \tag{A.28}$$

3. It may be useful to give another (equivalent) definition of Q which makes the analogy with the discussion in Sects.2,3 more transparent. Given the DeRham elliptic complex $(E_i, T_i), i = 0, 1, 2$ one can define the operator

$$P : E_0 \oplus E_2 \rightarrow E_1 \quad , \qquad P(\omega_0, \ \omega_2) = T_0 \omega_0 + T_1^* \omega_2 \quad ,$$

which maps a scalar and a 2-form into a vector. Then

$$P^\dagger : E_1 \rightarrow E_0 \oplus E_2 \quad , \qquad P^\dagger \omega_1 = (T_0^* \omega_1, \ T_1 \omega_1) \quad ,$$

$$P^\dagger P : E_0 \oplus E_2 \rightarrow E_0 \oplus E_2 \quad , \qquad P^\dagger P(\omega_0, \ \omega_2) = (T_0^* T_0 \omega_0, \ T_1 T_1^* \omega_2) \quad ,$$

$$P P^\dagger = \Delta_1 = T_0 T_0^* + T_1^* T_1 \quad .$$

Since $T_1 T_0 = 0$, $T_0^* T_1^* = 0$ the operator $P^\dagger P$ is diagonal in $E_0 \oplus E_2$ and $\det \ P^\dagger P = \det \ P P^\dagger = \det \ \Delta_1$. One can obtain a new non-diagonal elliptic second order operator on $E_0 \oplus E_2$ by introducing a "twist" in the definition of P,

$$P(\omega_0, \ \omega_2) = T_0 \omega_0 + A_1 T_1^* A_2 \omega_2 \quad , \tag{A.29}$$

where A_i are self-adjoint operators in E_i (e.g. multiplication by a function). In this case

$$Q \equiv P^\dagger P \quad , \quad Q(\omega_0,\ \omega_2) = (T_0^* T_0 \omega_0 + T_0^* A_1 T_1^* A_2 \omega_2,\ A_2 T_1 A_1^2 T_1^* A_2 \omega_2 + A_2 T_1 A_1 T_0 \omega_0) \quad . \tag{A.30}$$

Using the relation (17) between T_i and the exterior differentials

$$T_0 = d_0 \quad , \quad T_1 = d_1 \quad , \quad T_0^* = M_0^{-1} d_0^* M_1 \quad , \quad T_1^* = M_1^{-1} d_1^* M_2 \quad ,$$

and comparing (A.4),(A.7) with (A.29),(A.30) we conclude that they are equivalent if

$$A_1 = M_1 \quad , \quad A_2 = M_2^{-1} \quad , \quad M_0 = M_2 \quad ,$$

and if ω_0 is identified with ξ^1 and the 2-form ω_2 – with the scalar ξ^2. Explicitly,

$$Q = \begin{pmatrix} M_0^{-1} d_0^* M_1 d_0 & M_0^{-1} d_0^* M_1 d_1^* \\ M_0^{-1} d_1 M_1 d_0 & M_0^{-1} d_1 M_1 d_1^* \end{pmatrix} \quad , \tag{A.31}$$

and

$$\tilde{Q} = M_1 P P^\dagger M_1^{-1} = M_1 d_1^* M_0^{-1} d_1 + M_1 d_0 M_0^{-1} d_0^* \quad . \tag{A.32}$$

Eq.(A.32) is equivalent to (A.11).

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